Stochastic Processes

Midterm solution

2015-2016

Question 1:

1. If A and B are independent events, prove that A^c and B^c are independent.

$$P(A^c \cap B^c) = P(A \cup B)^c = 1 - P(A \cup B)$$

But $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Then $P(A^c \cap B^c) = 1 - \big(P(A) + P(B) - P(A \cap B)\big)$

But A and B are independent

$P(A \cap B) = P(A)P(B)$

Then $P(A^c \cap B^c) = 1 - \big(P(A) + P(B) - P(A)P(B)\big)$

$$P(A^c \cap B^c) = \big(1 - P(A)\big) - P(B) + P(A)P(B)$$
$$P(A^c \cap B^c) = \big(1 - P(A)\big) - P(B)\big(1 - P(A)\big)$$
$$P(A^c \cap B^c) = P(A^C) - P(B)P(A^C)$$
$$P(A^c \cap B^c) = P(A^C)\big(1 - P(B)\big)$$
$$P(A^c \cap B^c) = P(A^C)P(B^C)$$

Then A^c and B^c are independent.

2. Let A and B be events with P(A) = 1/2 , P(B) = 1/3, P(A⋂B) = 1/4.
   Find : P(A|B)   P(B|A)   P(A⋃B) $P(A^C|B^C)$ $P(B^C|A^C)$

i) P(A|B) = P(A⋂B)/P(B)
= (1/4) / (1/3)
= 3/4

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ii) P(B|A) = P(A⋂B)/P(A)
= (1/4) / (1/2)
= 1/2

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iii) P(A⋃B) = P(A)+P(B)-P(A⋂B)
= (1/2)+(1/3)-(1/4)
= 7/12

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iv) $P(A^C|B^C) = P(A^C \cap B^C)/P(B^C)$
$= P(A \cup B)^C / P(B^C)$
= (1- P(A ∪ B)) /(1-P(B))
= (1- (7/12)) /(1-(1/3))
= 5/8

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iv) $P(B^C|A^C) = P(A^C \cap B^C)/P(A^C)$
$= P(A \cup B)^C / P(A^C)$
= (1- P(A ∪ B)) /(1-P(A))
= (1- (7/12)) /(1-(1/2))
= 5/6

3. If x is a continuous random variable with the probability

   P(x) = Kx $0 \leqslant x \leqslant 2$ and zero elsewhere.

   Find the cummulative distribution function, mean, variance and standard deviation.

- Value of K:

  P(x) is a distribution function

  then $\int_{-\infty}^{\infty} P(x) = 1$

  $\int_{0}^{2} Kx = 1$

  $\frac{K}{2} x^2 \Big|_0^2 = 1$

  $\frac{K}{2} 2^2 = 1$

  $K = 1/2$

- Then P(x) = x/2

- Cummulative distribution function

  for $-\infty < x \leq 0$ $F(x) = 0$

  for $0 \leq x \leq 2$ $F(x) = \int_0^x \frac{1}{2} x$

  $F(x) = \frac{1}{4} x^2$

  for $2 \leq x < \infty$ $F(x) = 1$

- Mean

  $\mu = E(x)$

  $\mu = \int_{-\infty}^{\infty} x\, P(x)$

  $\mu = \int_0^2 \frac{1}{2} x^2$

  $\mu = \frac{1}{6} x^3 \Big|_0^2$

  $\mu = \frac{1}{6} 2^3$

  $\mu = \frac{4}{3} = 1.333$

- Variance

  $var(x) = E(x^2) - \mu^2$

  $var(x) = \int_{-\infty}^{\infty} x^2\, P(x) - \left(\frac{4}{3}\right)^2$

  $var(x) = \int_0^2 \frac{1}{2} x^3 - \left(\frac{4}{3}\right)^2$

  $var(x) = \frac{1}{8} x^4 \Big|_0^2 - \left(\frac{4}{3}\right)^2$

  $var(x) = \frac{1}{8} 2^4 - \left(\frac{4}{3}\right)^2$

  $var(x) = 2 - \frac{16}{9}$

  $var(x) = \frac{2}{9} = 0.222$

- Standard deviation σ

  $\sigma = \sqrt{var(x)} = \sqrt{\frac{2}{9}} = 0.471$

4. Given a and b are constants, Find with prove

   E(a) = ?   Var(aX +b) = ?

   where X is a continuous random variable

i) $E(a) = \int_{-\infty}^{\infty} a\, P(X)$

$E(a) = a \int_{-\infty}^{\infty} P(X)$ But $\int_{-\infty}^{\infty} P(X) = 1$

$E(a) = a$

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ii) $Var(aX + b) = Var(aX) + Var(b)$

But $Var(b) = 0$

and $Var(aX) = a^2\, Var(X)$

Then $Var(aX + b) = a^2\, Var(X)$

Question 2:

1. Three light bulbs are chosen at random from 20 bulbs of which 5 are defective. Find probability that:

   i- exactly one is defective,   ii- none is defective

   iii- at least one is defective   iv- at most one is defective.

i) exactly one is defective

P(1d) = P(1;3;5/20)

$= C_1^3 \left(\frac{5}{20}\right)^{1} \left(\frac{15}{20}\right)^{2}$

$= \frac{27}{64}$

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ii) none is defective

P(0d) = P(0;3;5/20)

$= C_0^3 \left(\frac{5}{20}\right)^{0} \left(\frac{15}{20}\right)^{3}$

$= \frac{27}{64}$

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iii) at least one is defective (E1)

P(E1) = P(1d) + P(2d) + P(3d)

= 1 - P(0d)

$= 1 - \frac{27}{64}$

$= \frac{37}{64}$

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iv) at most one is defective (E2)

P(E2) = P(0d) + P(1d)

$= \frac{27}{64} + \frac{27}{64}$

$= \frac{54}{64}$

2. Let X be a continuous random variable with distribution:

   f(x) = K (2-x) $0 \leqslant x \leqslant 2$ and zero elsewhere

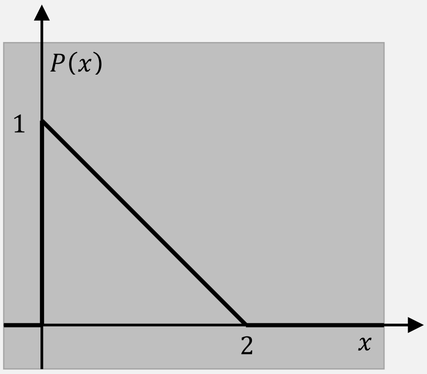
   Sketch the graph of f(x) and

   i- Evaluate K ii- Find $P(1 \leqslant x \leqslant 2)$

➢ Value of K:

P(x) is a distribution function

then $\int_{-\infty}^{\infty} P(x) = 1$



$$\int_0^2 K(2-x) = 1$$

$$-\frac{K}{2}(2-x)^2 \Big|_0^2 = 1$$

$$\frac{K}{2}(2-x)^2 \Big|_2^0 = 1$$

$$\frac{K}{2}(2-0)^2 = 1$$

$$K = 1/2$$

Then P(x) = (2 - x)/2

➢ $P(1 \leqslant x \leqslant 2)$

$$P(1 \leqslant x \leqslant 2) = \int_1^2 \frac{1}{2}(2-x)$$

$$P(1 \leqslant x \leqslant 2) = -\frac{1}{4}(2-x)^2 \Big|_1^2$$

$$P(1 \leqslant x \leqslant 2) = \frac{1}{4}(2-1)^2$$

$$P(1 \leqslant x \leqslant 2) = \frac{1}{4}$$

3. Let X be a random variable with the binomial P(k;n,p)

   Prove E(X) = np.

P(x) = P(k;n,p) $and\ q = 1 - p$

$$= \binom{n}{k}(p)^k(q)^{n-k}$$

$$E(x) = \sum_{k=0}^{n} k\binom{n}{k}(p)^k(q)^{n-k}$$

$$E(x) = (0)\binom{n}{0}(p)^0(q)^{n-0} + \sum_{k=1}^{n} k\binom{n}{k}(p)^k(q)^{n-k}$$

$$E(x) = \sum_{k=1}^{n} k\frac{n}{k}\binom{n-1}{k-1}(p)^k(q)^{n-k}$$

$$E(x) = np\sum_{k=1}^{n}\binom{n-1}{k-1}(p)^{(k-1)}(q)^{(n-1)-(k-1)}$$

Let $S = n-1, M = k-1$

when k changes from 1 to n

then M changes from 0 to n-1

then M changes from 0 to S

Then $E(x) = np\sum_{M=0}^{S}\binom{S}{M}(p)^M(q)^{S-M}$

$$E(x) = np(p+q)^S$$

$$E(x) = np(1)^S$$

Then $E(x) = np$

4. A fair die is tossed. Let X denotes twice the number appearing, and let Y denote 1 or 2 according as an odd or an even number appears. Find the probability, expectation, variance and standard deviation of X,Y

i) For X denotes twice the number appearing

die has values that appear from 1 to 6

So X changes from 2 to 12

and for example when $X = 6$ then 3 appears

- The Probability of X

X	2	4	6	8	10	12
P(X)	1/6	1/6	1/6	1/6	1/6	1/6

- Expectation E(X)

$$E(X) = \sum_{-\infty}^{\infty} XP(X)$$
$$E(X) = 7$$

- Variance Var(X)

$$Var(X) = E(X^2) - \left(E(X)\right)^2$$
$$Var(X) = \sum_{-\infty}^{\infty} X^2P(X) - (7)^2$$
$$Var(X) = \frac{182}{3} - (7)^2$$
$$Var(X) = \frac{35}{3} = 11.667$$

- Standard deviation σ

$$\sigma = \sqrt{Var(X)} = 3.42$$

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ii) For Y denote 1 or 2 according as an odd or an even number appears

die has values that appear from 1 to 6

So when $Y = 1$ then 1,3,5 appear

when $Y = 2$ then 2,4,6 appear

- The Probability of Y

Y	1	2
P(Y)	1/2	1/2

- Expectation E(Y)

$$E(Y) = \sum_{-\infty}^{\infty} YP(Y)$$
$$E(Y) = \frac{3}{2} = 1.5$$

- Variance Var(Y)

$$Var(Y) = E(Y^2) - \left(E(Y)\right)^2$$
$$Var(Y) = \sum_{-\infty}^{\infty} Y^2P(Y) - (1.5)^2$$
$$Var(Y) = \frac{5}{2} - (1.5)^2$$
$$Var(Y) = \frac{1}{4} = 0.25$$

- Standard deviation σ

$$\sigma = \sqrt{Var(Y)} = 0.5$$